Periods and the multiple gamma function in the archimedean case

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Colmez:
http://www.ams.org.proxy.lib.umich.edu/mathscinet/search/publdoc.html?pg1=MR&s1=1247996&loc=fromreflist

Chowla-Selberg
http://www.ams.org.proxy.lib.umich.edu/mathscinet/search/publdoc.html?pg1=MR&s1=0215797&loc=fromreflist

Chowla-Selberg formula:

K imag. quad fld. $\quad -d = $ discriminant

$\chi \longleftrightarrow K/\mathbb{Q}$

$$\exp\left(\frac{L'(0,\chi)}{L(0,\chi)}\right) \underset{\uparrow}{=} \frac{1}{d}\prod_{a=1}^{d-1} \Gamma\left(\frac{a}{d}\right)^{\frac{w\chi(a)}{2h}} \underset{\uparrow}{\sim} \pi\, \underbrace{P_K(id, id)^2}$$

Hurwitz-Lerch $\quad$ C-S. $\quad$ Shimura period symbol

$(a \sim b \Leftrightarrow b \neq 0, \ a/b \in \overline{\mathbb{Q}})$

$\pi P_K(id, id)$ period of $E/\overline{\mathbb{Q}}$ CM by K

§1 CM Periods

K: CM-fld $\quad [K:\mathbb{Q}] = 2n$

$J_K = \mathrm{Hom}(K, \mathbb{C}) \quad \rho$: cx conj.

$\Phi \subset J_K$ is a CM-type of $K \underset{\text{def.}}{\Longleftrightarrow} J_K = \Phi \sqcup \Phi\rho$

$A/\mathbb{C}$ Abelian variety, dim n

$\theta: K \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$

Rep. K on the space of holomorphic 1-form on $A \Rightarrow \Phi$: CM-type.

A: type (K, Φ) $\qquad$ A has CM by K through Φ.

$A/\overline{\mathbb{Q}}$, $\quad \sigma \in \Phi$

$\exists\, \omega_\sigma \neq 0, \quad a^*\omega_\sigma = a^\sigma \omega_\sigma \qquad \omega_\sigma$: holomorphic 1-form $\omega_\sigma/\overline{\mathbb{Q}}$.

$\exists\, P_K(\sigma, \Phi) \in \mathbb{C}^\times$ s.t.

$$\int_c \omega_\sigma \sim \pi P_K(\sigma, \Phi) \qquad \forall\, c \in H_1(A, \mathbb{Z}).$$

$P_K(\sigma, \Phi)$ mod $\overline{\mathbb{Q}}^\times$, independent of ω_σ, A.

Thm S1 (Shimura)

Thm 5.1 (Shimura)

$I_K = \mathbb{Z}^{J_K}$ = the free abelian gp generated by J_K

$\exists\ p_K : I_K \times I_K \longrightarrow \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$ s.t.

(1) $p_K(\sigma, \Phi)$ is defined as above

(2) p_K is bilinear $\qquad p_K(\xi_1 + \xi_2, \eta) = p_K(\xi_1, \eta)\, p_K(\xi_2, \eta)$.

(3) $p_K(\xi, \eta\rho) = p_K(\xi\rho, \eta) = p_K(\xi, \eta)^{-1}$

(4) $K \subset L$

$$p_K(\xi, \mathrm{Res}_{L/K}(\zeta)) = p_L(\mathrm{Inf}_{L/K}(\xi), \zeta)$$

$$p_K(\mathrm{Res}_{L/K}(\zeta), \xi) = p_L(\zeta, \mathrm{Inf}_{L/K}(\xi))$$

$\begin{matrix} L \\ | \\ K \end{matrix}$ (5) $\gamma : K' \simeq K$

$$p_{K'}(\gamma\xi, \gamma\eta) = p_K(\underbrace{\xi, \eta}_{\in I_K})$$

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Ex (Weil) $\quad \ell$ odd prime $\quad 1 \leq a \leq \frac{(\ell-1)}{2}$

$C: \; y^\ell = x^a(1-x)$

$g(C) = (\ell-1)/2$

$J = \mathrm{Jac}(C) \quad : \quad$ type $(K, \Phi_a) \qquad K = \mathbb{Q}(\underbrace{e^{2\pi i/\ell}}_{= \zeta_\ell})$

$T_a = \{ t \mid 1 \leq t \leq \ell-1, \; \langle \frac{at}{\ell} \rangle + \langle \frac{t}{\ell} \rangle < 1 \}$

fractional part

$\Phi_a = \{ \sigma(t) \mid t \in T_a \}$

$\zeta_\ell \to \zeta_\ell^t$

beta function

$\pi P_K(\sigma(t), \Phi_a) \sim B(\langle \frac{at}{\ell} \rangle, \langle \frac{t}{\ell} \rangle) \qquad t \in T_a.$

Take $\ell = 7, \quad a = 2 \qquad T_a = \{1, 2, 4\}$

Put $K_0 = \mathbb{Q}(\sqrt{-7}) \subset K$

$P_{K_0}(\mathrm{id}, \mathrm{id}) \sim P_K(\mathrm{id}, \mathrm{Inf}_{K/K_0}(\mathrm{id}))$

$\sim P_K(\mathrm{id}, \sigma(1) + \sigma(2) + \sigma(4)) \sim \pi^{-1} \frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})}{\Gamma(\frac{3}{7})} \qquad \Gamma(s)\Gamma(-s) = \frac{\pi}{\sin \pi s}$

$\Downarrow$

Chowla-Selberg for K_0

Thm 1 (Anderson): $\quad K \quad$ CM-fld, abelian over $\mathbb{Q}$

$G = \mathrm{Gal}(K/\mathbb{Q})$.

gamma fcn

$P_K(\mathrm{id}, \tau) \sim \pi^{-\mu(\tau)/2} \prod_{\omega \in \hat{G}_-} \exp\left(\frac{\omega(\tau)}{|G|} \frac{L'(0,\omega)}{L(0,\omega)} \right) \quad , \tau \in G.$

$\mu(\tau) = \begin{cases} 1 & \tau = 1 \\ -1 & \tau = \rho \\ 0 & \tau \neq 1, \rho \end{cases} \qquad \hat{G}_- : \omega(\rho) = -1.$

Conj A: (Colmez, Y.)

$K = $ CM fld, normal over $\mathbb{Q}$ $\quad G = \mathrm{Gal}(K/\mathbb{Q})$ $\quad c$: a conj. class in G

$$\prod_{\tau \in C} p_K(\mathrm{id}, \tau) \sim \pi^{-\mu(c)/2} \prod_{\omega \in \hat{G}_-} \exp\left(\frac{|c|\chi_\omega(c)}{|G|}, \frac{L'(0,\omega)}{L(0,\omega)} \right)$$

$$\mu(c) = \begin{cases} 1 & c = \{1\} \\ -1 & c = \{\rho\} \\ 0 & c \neq \{1\}, \{\rho\} \end{cases} \qquad \chi_\omega = \mathrm{tr} \text{ of } \omega. \qquad \rho \in Z(G)$$

<u>Thm</u> 2: $K = $ CM fld, abelian over $F = $ totally real

Put $J_F = \{\sigma_1, \dots, \sigma_n\}$ $\quad G = \mathrm{Gal}(K/F)$

Assume Conj. A.

then $\sigma_i \to \sigma_i \in J_K$. $\qquad \tau \in G$.

$$\prod_{i=1}^{n} P_K(\sigma_i, \tau\sigma_i) \sim \pi^{-n\mu(\tau)/2} \prod_{\omega \in \hat{G}_-} \exp\left(\frac{\omega(\tau)}{|G|} \frac{L'(0,\omega)}{L(0,\omega)} \right)$$

$$\parallel$$

$$\prod_{i=1}^{n} P_{K^{\sigma_i}}(\mathrm{id}, \sigma_i^{-1}\tau\sigma_i)$$

—.

$K = $ CM fld $\qquad \mathfrak{q} \subseteq \mathcal{O}_K \qquad \psi : I_{\mathfrak{q}} \to \mathbb{C}^\times$

$$\psi((\alpha)) = \prod_{\sigma \in J_K} (\alpha^\sigma)^{l_\sigma} \qquad \alpha \equiv 1 \bmod^\times \mathfrak{q} \qquad l_\sigma \in \mathbb{Z}.$$

<u>Thm</u> S2 (Shimura)

Take a CM-type Φ so that $l_\sigma \leq l_{\sigma\rho}$, $\sigma \in \Phi$

If $m \in \mathbb{Z}$ satisfies $l_\sigma < m \leq l_{\sigma\rho}$, $\forall \sigma \in \Phi$

$$L(m, \psi) \sim \pi^e P_K\left(\sum_{\sigma \in \Phi} (l_{\sigma\rho} - l_\sigma)\sigma, \Phi \right) \qquad e = m\frac{[K:\mathbb{Q}]}{2} - \sum_{\sigma \in \Phi} l_\sigma$$

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§2 Multiple gamma function and Shintani's formula

$\omega = (\omega_1, \ldots, \omega_r) \qquad \omega_i > 0 \ , \quad x > 0$

$$\zeta_r(s, \omega, x) = \sum_{m_1, \ldots, m_r \geq 0} (x + m_1\omega_1 + \cdots + m_r\omega_r)^{-s}$$

$$\frac{\partial}{\partial s} \zeta_r(s, \omega, x)\Big|_{s=0} = \log \frac{\Gamma_r(x, \omega)}{\rho_r(\omega)} \qquad r\text{-ple gamma}$$

...

Stark-Shintani conjecture involving partial zeta functions...